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Exercise 3 – 09.10.2025 – Solution

Perfect plasticity

1. Parameter determination for two yield criteria

For the present case, where perfect plasticity is assumed, the ultimate state is defined as the point where the stress reaches its maximum value. The following notation is used to designate the ultimate state: $(...)_u$.
All equations and simplifications written in the following are valuable for the triaxial states.

Part 1

- **Figure 1**. Conventional Triaxial Compression test (CTC): $\sigma_3 = cte = \sigma_0 = 10kPa$.

$$\sqrt{J_{2D}} = \sqrt{\frac{1}{6}[(\sigma_1 - \sigma_3)^2 + (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2]}$$

For triaxial tests $\sqrt{J_{2D}} = \sqrt{\frac{1}{3}}(\sigma_1 - \sigma_3)$

$$\sigma_1 - \sigma_3 = 40\ kPa$$

$$\sqrt{J_{2D}} = 23.1\ kPa$$

$$\sigma_1 = \sigma_a = 50\ kPa, \sigma_3 = \sigma_r = 10\ kPa$$

$$J_1 = \sigma_1 + \sigma_2 + \sigma_3 = \sigma_a + 2\sigma_r = 70kPa$$

- **Figure 2.** Conventional Triaxial Compression test (CTC): $\sigma_3 = cte = \sigma_0 = 20kPa$.

$$\sqrt{J_{2D}} = 40.4\ kPa$$

$$\sigma_1 - \sigma_3 = \sqrt{3}\sqrt{J_{2D}} = 70kPa$$

$$\sigma_1 = \sigma_a = 90\ kPa, \sigma_3 = \sigma_r = 20\ kPa$$

$$J_1 = 130\ kPa$$

- **Figure 3.** Reduced Triaxial Extension (RTE): $\sigma_3 = cte = \sigma_0 = 20kPa$

In the case of the RTE test: σ_a becomes inferior to σ_r, so: $\sigma_a = \sigma_3$ and $\sigma_r = \sigma_1 = 20\ kPa$

$$\sigma_1 - \sigma_3 = \sigma_r - \sigma_a = 19\ kPa$$

$$\sigma_a = 1\ kPa$$

$$\sigma_1 - \sigma_3 = 19\ kPa = \sqrt{3}\sqrt{J_{2D}}$$

Thus,

$$\sqrt{J_{2D}} = 11\ kPa$$

$$J_1 = 41\ kPa$$

Table 1 summarizes the ultimate values of σ_a and σ_r, their combinations in terms of τ_{max}, σ_m, q, p', and the invariants J_1 and $(J_{2D})^{0.5}$

Figure	σ_a	σ_r	τ_{max}	σ_m	q	p'	$(J_{2D})^{0.5}$	J_1
1	50	10	20	30	40	23.3	23.1	70
2	90	20	35	55	70	43.3	40.4	130
3	1	20	9.5	10.5	-19	13.7	11.0	41

Table 1. Ultimate stresses values and invariants from Figure 1,2,3

Resulting plots:

- Figure A shows the results of the tests in the plane (σ_m - τ_{max}) for the determination of the Mohr-Coulomb parameters (c' and φ'). Only the results of the two compression tests are used.
  In the plane (σ_m - τ_{max}), the corresponding equation of MC is:

  $$\tau_{max} = c' \cdot \cos\varphi' + \sigma_m \sin\varphi'$$

  Where, according to figure A → $c' \cdot \cos\varphi' = 2.0$ and $\sin\varphi' = 0.6$

  $\varphi' = 37°$

  $c' = 2.5\ \mathrm{kPa}$

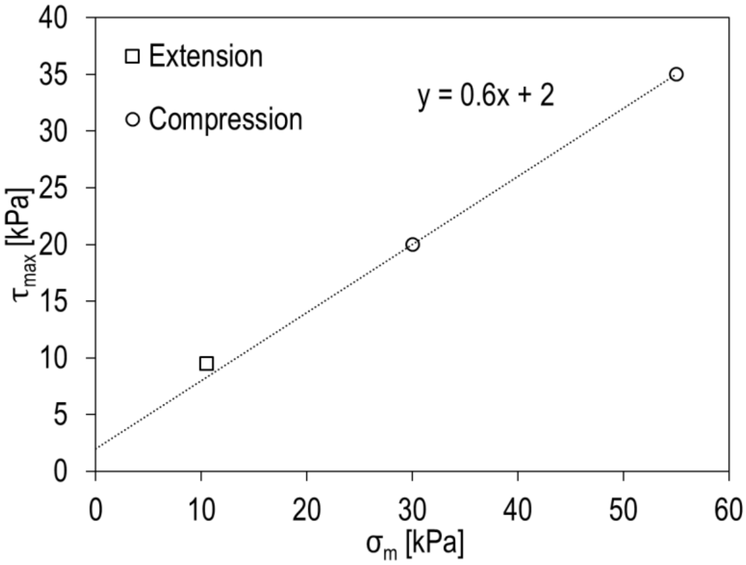

Figure A. Determination of the Mohr-Coulomb parameters

Notice that for the determination of the M-C failure criterion in compression (line interpolation in Figure A), only the two CTC tests (circles) can be used; the result of the RTE cannot be used as the Mohr-Coulomb foresees different strength in compression and extension. Since one point is not sufficient for the definition of the criterion in extension, this value has to be complemented with other extension tests.

- Figure B shows the tests results in the plane ($J_1 - \sqrt{J_{2D}}$) for the determination of the Drucker-Prager (D-P) parameters (k and α). Note that D-P criterion is the same in both compression and extension; hence, all points obtained from the three tests can be used for its determination.
  In the plane ($J_1 - \sqrt{J_{2D}}$), the equation of the D-P failure criterion is:

  $$\sqrt{J_{2D}} = \alpha J_1 + k$$

When fitting the experimental results, it is important to respect the limitation on the minimum value of k ($k \geq 0$). Figure B shows the experimental points (circles for CTC tests and square for RTE test) as well as the failure line from which the two parameters can be obtained.

$$\mathrm{k} = 0$$
$$\alpha = 0.31$$

With the values of k and α, it is possible to calculate the values of φ' and c' according to the following equations:

$$\alpha = \frac{2\sin\varphi'}{\sqrt{3}(3-\sin\varphi')} 0.31, \quad k = \frac{6\,c'\cos\varphi'}{\sqrt{3}(3-\sin\varphi')} = 0$$

from which we obtain φ'=39° and c'=0 kPa.

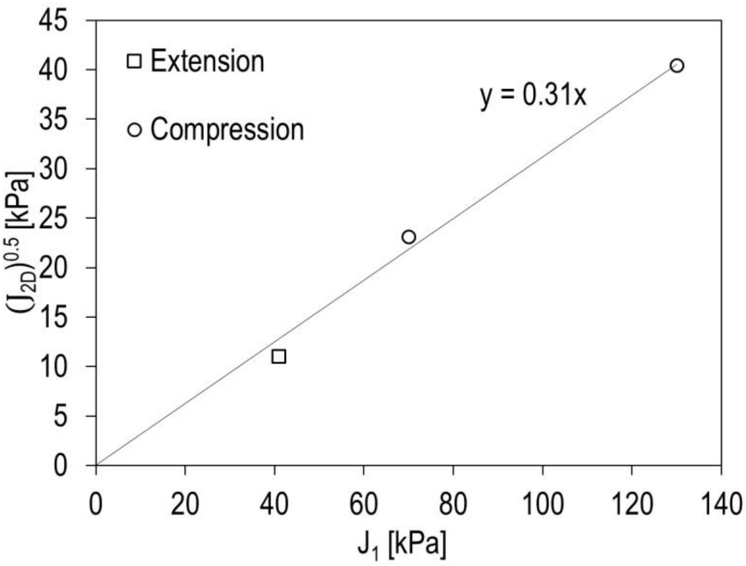

Figure B. Determination of Drucker-Prager parameters

Part 2

- **Figure 4**. Reduced Triaxial Extension (RTE): $\sigma_3 = cte = \sigma_0 = 10kPa$

The calculation procedure used for figure 3 are applicable to the analysis of figure 4.

$$\sqrt{J_{2D}} = 5.5\ kPa$$

$$\sigma_1 - \sigma_3 = \sqrt{3}\sqrt{J_{2D}} = 9.5\ kPa$$

$$\sigma_r = \sigma_1 = 10\ kPa, \sigma_a = \sigma_3 = 0.5\ kPa$$

$$J_1 = 20.5\ kPa$$

- **Figure 5**. Conventional Triaxial Extension (CTE): $\sigma_1 = cte = \sigma_0 = 20kPa$.

$$\sqrt{J_{2D}} = 20.5\ kPa$$

$\sigma_1 - \sigma_3 = \sqrt{3}\sqrt{J_{2D}} = 35.5\ kPa$

$\sigma_r = \sigma_1 = 55.5\ kPa, \sigma_a = \sigma_3 = 20\ kPa$

$J_1 = 131\ kPa$

- **Figure 6**. Triaxial Compression (TC): $\sigma_3 = \sigma_0$ = 25 kPa and it is not constant during the test

During the compression stage, the stress state remains in the octahedral plane (meaning that mean stress J_1 is constant)

$J_1 = \sigma_1 + \sigma_2 + \sigma_3 = 3\sigma_0$

Figure 6 indicates $\sqrt{J_{2D}} = 27.3\ kPa$, $\sigma_1 - \sigma_3 = \sqrt{3}\sqrt{J_{2D}} = 47.25\ kPa$

Here, $\sigma_1 = \sigma_a$ and $\sigma_2 = \sigma_3 = \sigma_r$

For triaxial compression (TC), the mean stress remains constant (i.e., $\Delta J_1 = 0$)

Thus, $J_1 = \sigma_a + 2\sigma_r = 75\ kPa$

From $\sigma_a - \sigma_r = 47.25\ kPa$ and $\sigma_a + 2\sigma_r = 75\ kPa$, we get $\sigma_a = 56.5\ kPa, \sigma_r = 9.25\ kPa$

- **Figure 7**. Triaxial Extension (TE): σ_0 = 10 kPa and it is not constant during the test

The interpretation is similar to that of figure 6. In this case for an extension test:

$\sqrt{J_{2D}} = 5.20\ kPa$

$\sigma_1 - \sigma_3 = \sqrt{3}\sqrt{J_{2D}} = 9.0\ kPa$

$\sigma_r = \sigma_1 = 13.0\ kPa, \sigma_a = \sigma_3 = 4.0\ kPa$

$J_1 = 30\ kPa$

Resulting plots:

Figures A and B can be completed with the new numerical results obtained in Part 2. The plots in figures C and D are obtained.

- Figure C shows the tests results in the plane (σ_m -τ_{max}) for the determination of the Mohr-Coulomb parameters (c' and φ'). In this plane, the corresponding equation is:

  $\tau_{\max} = c' \cdot \cos\varphi' + \sigma_m \sin\varphi'$

  As now there are sufficient tests in extension and compression, it is possible to determine the parameters in both cases.

  The M-C failure criterion in compression gives: → $c' \cdot \cos\varphi' = 3.91$ and $\sin\varphi' = 0.57$

  $\varphi' = 34.8°$

  $c' = 4.8\ \text{kPa}$

  The M-C failure criterion in extension gives: → $c' \cdot \cos\varphi' = 2.98$ and $\sin\varphi' = 0.40$

  $\varphi' = 23.6°$

  $c' = 3.2\ \text{kPa}$

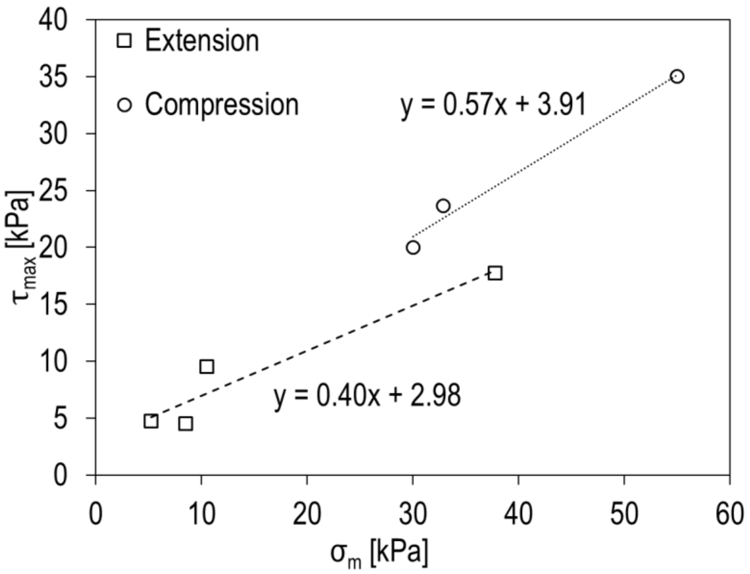

Figure C. Determination of Mohr-Coulomb parameters

- Figure D reports the tests results in the plane ($J_1 - \sqrt{J_{2D}}$) for the determination of the Drucker-Prager parameters (k and α). Since the D-P criterion is the same in both compression and extension, all points from the seven tests are used for its determination (without distinction between compression and extension). In the plane ($J_1 - \sqrt{J_{2D}}$), the equation of the Drucker-Prager failure criterion is:

  $$\sqrt{J_{2D}} = \alpha J_1 + k$$

  When fitting the experimental results, it is important to respect the limitation on the minimum value of k (k≥0). Figure D shows the experimental points (circles for CTC tests and square for RTE test) as well as the failure line from which the two parameters can be obtained.

  $$\mathrm{k} = 2.32$$
  $$\alpha = 0.23$$

  Given the value of k and α, it is possible to calculate the value of φ' and c' according to the following equation:

  $$\alpha = \frac{2\sin\varphi'}{\sqrt{3}(3-\sin\varphi')}0.23, \quad k = \frac{6\,c'\cos\varphi'}{\sqrt{3}(3-\sin\varphi')} = 2.32$$

  from which we obtain φ'=30° and c'=1.9 kPa.

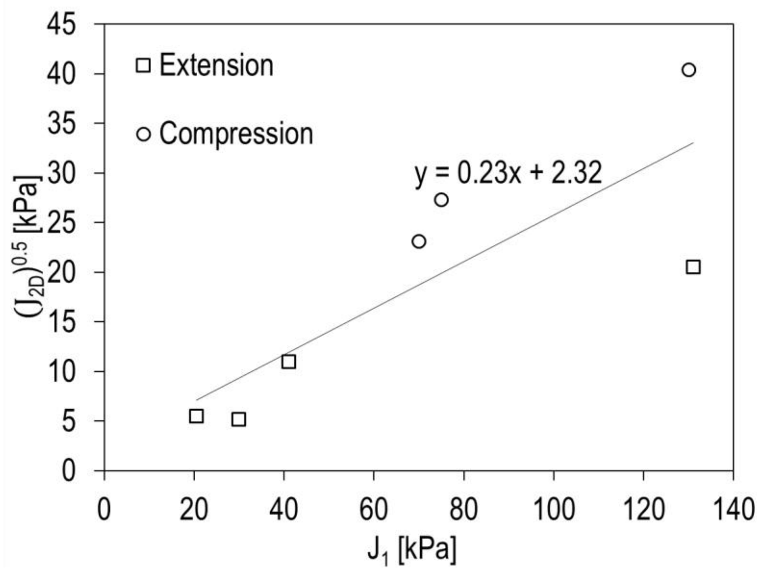

Figure D. Determination of Drucker-Prager parameters

Figure	σ_a	σ_r	τ_{max}	σ_m	q	p'	$(J_{2D})^{0.5}$	J_1
1	50	10	20	30	40	23.3	23.1	70
2	90	20	35	55	70	43.3	40.4	130
3	1	20	9.5	10.5	-19	13.7	11	41
4	0.5	10	4.75	5.25	-9.50	6.83	5.48	20.5
5	20	55.5	17.75	37.75	-35.50	43.67	20.50	131
6	56.5	9.25	23.63	32.88	47.25	25.00	27.28	75
7	4	13	4.50	8.50	-9.00	10.00	5.20	30

Table 2. Summary of ultimate stresses values and invariants from Part 1 and Part 2 (stresses are expressed in kPa)

Model	Part 1 (figures A-B)	Part 2 (figures C-D)
Mohr Coulomb (compression)	c' = 2.5 kPa , $\varphi'=\sin^{-1}(0.6)=37°$	c' = 4.8 kPa , $\varphi'=\sin^{-1}(0.54)=34.8°$
Mohr Coulomb (extension)	-	c' = 3.2 kPa, $\varphi'=\sin^{-1}(0.42)=23.6°$
Drucker Prager	k = 0, α=0.31 c' = 0.0 kPa , φ'=39°	k = 2.32, α=0.23 c' = 1.9 kPa , φ'=30°

Table 3. Summary of parameters of the M-C and D-P failure criteria from Part 1 and part 2